The Microscopic Picture of Chiral Luttinger Liquid: Composite Fermion Theory of Edge States

Yue Yu, Wenjun Zheng and Zhongyuan Zhu
Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China

We derive a microscopic theory of the composite fermions describing the low-lying edge excitations in the fractional quantum Hall liquid. Using the composite fermion transformation, one finds that the edge states of the $\nu = 1/m$ system in a disc sample are described by, in one dimensional limit, the Calogero-Sutherland model with other interactions between the composite fermions as perturbations. It is shown that a large class of short-range interactions renormalize only the Fermi velocity while the exponent $g = \nu = 1/m$ is invariant under the condition of chirality. By taking the sharp edge potential into account, we obtain a microscopic justification of the chiral Luttinger liquid model of the fractional quantum Hall edge states. The approach applied to the $\nu = 1/m$ system can be generalized to the other edge states with odd denominator filling factors.

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I. INTRODUCTION

The study of edge excitations of fractional quantum Hall effect (FQHE) has evoked considerable interests in the past decade. Compared with the edge states of integer quantum Hall effect (IQHE), which are well understood to be a chiral Fermi liquid [1], the description of FQHE is a strongly correlated many-body problem. It is now believed that the bulk property of a FQH liquid can be described by an incompressible quantum liquid with only gapful excitations [2], while the gapless ones exist along the edge. In order to understand the edge excitations many attempts have been made [3–7]. From a macroscopic point of view, Wen suggests that the FQHE edge with $\nu = 1/m$ (m is an odd integer) is actually a chiral Luttinger liquid (CLL) with its characteristic exponent $g = \nu$ as a topological index which is invariant to perturbations like the interactions between electrons [6]. In deriving the macroscopic theory, one starts from a few general assumptions, such as the gauge invariance of the system and arrives at the macroscopic effective field theory without being concerned with the details of the microscopic interactions. However, a microscopic justification of the effective theory is still lacking, despite some recent efforts which make use of collective field method to give a random phase approximation solution to the equations of motion [7]. Recently, a mean-field consideration of the fermion Chern-Simons theory of the edge states has been presented [5]. Meanwhile, several authors [4] point out the possible relations between the edge states of FQHE and the Calogero-Sutherland model (CSM) [8].

Recently, two of us [9] give a derivation of a microscopic model of the composite fermion (CF) [10] type excitations at the FQHE edge with an odd denominator filling factor $\nu = 1/m$ and relates the model to the CSM in the one-dimensional (1-d) limit. In this paper, we present the details and some unpublished results ,and give a generalization to the other situations with odd denominator filling factors. We start from the general Hamiltonian for a two-dimensional system of interacting electrons in a strong perpendicular magnetic field and bound into some specific geometry like a disc. After a CF type of anyon [11] transformation [12–17], we obtain a Hamiltonian whose ground state wave function can be simply written out. By extracting the excitation part from the total wave function, we arrive at a form of Hamiltonian which is shown to be equivalent to the corresponding Hamiltonian of the CSM in the 1-d limit. We then take into account the influence of the interactions between CFs. By using the analog of the 1-d two-body Schrodinger equation to the wave equation satisfied by the radial wave function of a particle in a three-dimensional centrally symmetric field, we demonstrate that under the condition of chirality the topological exponent $g = 1/m$ is indeed robust against the perturbation of a class of short-range interactions like the δ-function interaction and the pseudopotentials [18] while the Fermi velocity can be altered. It is emphasized that the low-lying excitations of the CSM are described by the $c = 1$ conformal field theory (CFT) with its compactified radius $\mathcal{R} = 1/\sqrt{m}$ [19,20], which gives a profound reason for its low-energy fixed point of a Luttinger liquid. The bare Coulomb interaction will affect the exponent because of its infrared divergence. However, it is expected that the 1-d Coulomb interaction is actually drastically renormalized or partially screened by either the bulk electrons or the experimental devices such as a metal electrode. We consider the case of drastically renormalized Coulomb interaction and show that the exponent g is not changed while a branch of the 1-d plasmon excitations appears, which agrees with the CLL's results of the influence of the Coulomb interaction [21].

The chirality of the edge excitations is a basic assumption of the macroscopic effective theory like the CLL. However, the microscopic understanding of it is still elusive. Recently, Chklovskii and Halperin try to show it in the CF picture beyond the mean-field theory [22]. In this paper, we derive the chirality from the microscopic point of view in the framework of the present theory. Since the edge CFs are confined in a very narrow strip near

the boundary of the sample, the single particle picture will work for the radial wave functions except that the wave vectors of the CFs are strongly coupled through the asymptotic Bethe ansatz equations of the CSM. Furthermore, one finds that the left-moving sector of the excitations of the CSM is suppressed because the magnetic field breaks the left and right moving symmetry of the CSM. The sound wave velocity of the theory is renormalized after turning on an external electric field and the increment of it agrees with the drift velocity of CF in the effective magnetic field. However, Haldane's velocity relations of the Luttinger liquid [23] are invariant, which implies that the low-lying excitations of the system are still governed by the $c = 1$ CFT without changing the compactified radius . In this sense, we finally arrive at the justification of the CLL theory of the edge excitations for the $\nu = 1/m$ FQHE from the microscopic point of view.

The approach applied to the edge states with $\nu = 1/m$ can be generalized to the edge states with other odd denominator filling factors ,where a K-matrix, which is employed by Wen and Zee in the effective field theory of FQHE [24], is used to characterise the exponents of the multi-branch CLL.

This paper is organized as follows: In Sec. II, we give a detailed derivation of the microscopic model of the FQHE edge states with $\nu = 1/m$ and relate the model to the CSM whose low-lying excitations are governed by the $c = 1$ CFT. In Sec. III, we discuss the influences of the interactions between edge CFs and show that the topological exponent is not renormalized by some short-range interactions because of chirality while the Coulomb interaction leaves some more subtle effects to be clarified. In Sec. IV, the radial degree of freedom is included and we provide the microscopic justification of the CLL theory of FQHE edge modes. A generalization to other filling factors is presented in Sec. V. We conclude the paper in the final section.

II. DERIVATION OF THE MICROSCOPIC EDGE MODEL

We start from the following Hamiltonian which describes a system of two dimensional interacting electrons with their spins polarized by a strong magnetic field,

$$H_{\rm e} = \sum_{\alpha=1}^{N} \frac{1}{2m_b}[\vec{p}_\alpha + \frac{e}{c}\vec{A}(\vec{r}_\alpha)]^2 + \sum_{\alpha<\beta} V(\vec{r}_\alpha - \vec{r}_\beta) + \sum_\alpha U(\vec{r}_\alpha), \tag{2.1}$$

where $V(\vec{r})$ represents the interactions between electrons, which will be specified later. $U(\vec{r})$ is an external potential that serves as a confinement to hold the electrons in some area of certain geometry, say a disc of radius R as it is assumed throughout this paper. Furthermore, we assume the potential is sharp enough at the boundary of the sample to avoid complex structure of the edge states while it is smooth away from the edge such that it has no influences on the bulk of Hall liquid. m_b is the band mass of the electron and $\vec{A}$ is the vector potential providing an external magnetic field normal to the plane.

The composite particle transformation will bring us to a good starting point to deal with the FQHE as many successful investigations have told us [12–15]. In order to change into a CF picture, we can perform an anyon transformation which reads

$$\Psi_e(z_1, ..., z_N) = \prod_{\alpha<\beta}\left[\frac{z_\alpha - z_\beta}{|z_\alpha - z_\beta|}\right]^{\tilde{\phi}} \Psi_c(z_1, ..., z_N), \tag{2.2}$$

where Ψ_e gives the electronic wave function, while Ψ_c is the wave function of CFs. $\tilde{\phi}$ is the number of flux quanta attached to one electron, which is an even number for CF. After this transformation, we obtain the following Hamiltonian,

$$\begin{aligned} H_{\rm cf} = &\sum_{i=1}^{N}[\vec{p}_i + \vec{A}(\vec{r}_i) + \vec{a}(\vec{r}_i)]^2 \\ &+ \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U(\vec{r}_i), \end{aligned} \tag{2.3}$$

where

$$\vec{a}(\vec{r}_i) = \tilde{\phi}\sum_{j\neq i}\frac{\hat{z}\times(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}, \tag{2.4}$$

is a statistical gauge potential responsible for the bound flux. In this CF Hamiltonian, we have replaced the electron band mass m_b by a phenomenological effective mass m^* of the CF and use the unit $\hbar = 2m^* = e/c = 1$ in what follows.

This Hamiltonian can be rewritten in a more suggestive form which reads

$$\begin{aligned} H_c = &\sum_{i=1}^{N}\Big[(\frac{-i}{r_i}\frac{\partial}{\partial\phi_i} + \frac{\partial E_i'}{\partial r_i})^2 + (-i\frac{\partial}{\partial r_i} - \frac{1}{r_i}\frac{\partial E_i'}{\partial \phi_i})^2 \\ &- \frac{1}{r_i}\frac{\partial}{\partial r_i} + \frac{i}{r_i^2}\frac{\partial E_i'}{\partial\phi_i}\Big] + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U(\vec{r}_i), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} E_i &= \sum_j{}' E_{ij} \\ &= \tilde{\phi}\sum_j{}' \ln|\vec{r}_i - \vec{r}_j| \\ E_i' &= \sum_j{}' E_{ij}' \\ &= \sum_j{}'(E_{ij} - \frac{\beta r_i^2}{N-1}) \\ &= E_i - \beta r_i^2, \end{aligned} \tag{2.6}$$

and $\beta = \frac{|B|}{4}$. Here we suppose $B < 0$. This form reminds us of a two dimensional plasma with E_{ij} closely resembling the 2d Coulomb interaction potential. The analog has been very useful to numeric calculations of FQHE.

Next we conduct a non-unitary transformation of the Hamiltonian H_c as follows [16,17],

$$\Psi_c = \exp(\sum_{i<j} E'_{ij})\Phi \tag{2.7}$$

$$= \exp(-\sum_i \beta r_i^2) \prod_{i<j} |z_i - z_j|^{\tilde{\phi}}\Phi, \tag{2.8}$$

$$H' = \exp(\sum_{i<j} -E_{ij}{}')H_c \exp(\sum_{i<j} E_{ij}{}') \tag{2.9}$$

$$= -4\sum_{i=1}^{N} \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i} - \sum_{i=1}^{N}\{4\tilde{\phi}\sum_j{}' \frac{1}{z_i - z_j} - 2|B|\bar{z}_i\}\frac{\partial}{\partial \bar{z}_i}$$
$$+ \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U(\vec{r}_i).$$

Although the Hamiltonian H' so obtained is not Hermitian in itself, it does provide us with some insights of the ground state wave function. Obviously, any nonsingular complex function of z_i gives an eigenstate of H', which actually represents its ground state at least in case of the CF picture in the sense of mean field, that is

$$\Phi_g = f(z_1, ..z_N). \tag{2.10}$$

If we do not consider the interaction, the ground state is highly degenerate. However, the interaction breaks the degeneracy and the groundstate has to maintain a minimal angular momentum as Laughlin has suggested in the presentation of his famous wave functions. For the CF system, the ground state reads

$$f(z_1, ...z_N) = \prod_{i<j}(z_i - z_j). \tag{2.11}$$

Thus, we arrive at an effective Hamiltonian by expressing the wave function Φ as a product of the ground state wave function Φ_g and an extra part Φ'', and consider only the Hamiltonian corresponding to the latter, which is

$$H'' = \Phi_g^{-1} H' \Phi_g \tag{2.12}$$
$$= -4\sum_{i=1}^{i=N} \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i}$$
$$- \sum_{i=1}^{i=N}\{4m\sum_j{}' \frac{1}{z_i - z_j} - 2|B|\bar{z}_i\}\frac{\partial}{\partial \bar{z}_i}$$
$$+ \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U(\vec{r}_i)$$

If we reverse the above transformations from H'' to H_e, we will arrive at the ground state wave function

$$\Psi_{eg}(z_1, ...z_N) = \exp(-\sum_i \beta r_i^2) \prod_{i<j}(z_i - z_j)^m, \tag{2.13}$$

which is precisely the well-known wave function first proposed by Laughlin.

Based on the above understanding of the CF groundstate wave function, we now turn to the edge states theory of the CF excitations. The partition function of the system is given by

$$Z = \sum_{N^e} C_N^{N^e} \int_\partial d^2z_1....d^2z_{N^e} \int_B d^2z_{N^e+1}...d^2z_N \tag{2.14}$$
$$\times \left(\sum_\delta |\Psi_\delta|^2 e^{-\beta(E_\delta + E_g)} + \sum_\gamma |\Psi_\gamma|^2 e^{-\beta(E_\gamma+E_g)}\right),$$

where we have divided the sample into the edge ∂ and the bulk B. E_g is the ground state energy and E_δ are the low-lying gapless excitation energies with δ being the excitation branch index. E_γ are the gapful excitation energies. At $\nu = 1/\tilde{\phi}$, the low-lying excitations are everywhere in the sample and we do not consider this case here. We are interested in the case $\nu = \frac{1}{\tilde{\phi}+1} = 1/m$, where the bulk states are gapful. The low-lying excitations are confined in the edge of the sample. For convenience, we consider a disc geometry sample here. The edge potential is postulated with a sharp shape. The advantage of the CF picture is we have a manifestation that the FQHE of the electrons in the external field B could be understood as the IQHE of the CFs in the effective field B^* defined by $B^*\nu^* = B\nu$. For the present case, $B^* = B/m$ and $\nu^* = 1$. The energy gap in the bulk is of the order $\hbar\omega_c^*$ with the effective cyclotron frequency $\omega_c^* = \frac{eB^*}{m^*c}$ (m^* is the effective mass of the CF). Hereafter, we use the unit $\hbar = e/c = 2m^* = 1$ except the explicit expressions. By the construction of the CF, the FQHE of the electrons can be described by the IQHE of the CFs [10] while the electrons in the $\nu = 1/\tilde{\phi}$ field could be thought as the CFs in a zero effective field. Thus, a Fermi-liquid like theory could be used [15] and we have a set of CF-type quasiparticles. Applying the single particle picture, which Halperin used to analyze the edge excitations of the IQHE of the electrons, to the edge excitations of the CFs, one could have a microscopic theory of the quasiparticles at the edge. In the low-temperature limit, the domination states contributing to the partition function are those states that the lowest Landau level of the CF-type excitations is fully filled in the bulk but only allow the edge CF-type excitations to be gapless because the gap is shrinked in the edge due to the sharp edge potential. The other states with their energy $E_\gamma + E_g$ open a gap at least in the order of $\hbar\omega_c^*$ to the ground state. In the low-temperature limit, $k_BT \ll \hbar\omega_c^*$, the effective partition function is

$$Z \simeq \sum_{\delta,N^e} C_N^{N^e} \int_\partial d^2z_1...d^2z_{N^e}|\Psi_{e,\delta}|^2 e^{-\beta(E_\delta(N^e)+E_{g,b})} \tag{2.15}$$

$$= \sum_{N^e} C_N^{N^e} \mathrm{Tr}_{(\mathrm{edge})} e^{-\beta(H_e+E_{g,b})},$$

where the trace runs over the low-lying set of the quantum state space for a fixed N_e and, according to the single particle picture, $\Psi_{e,\delta}$ are the edge many-quasiparticle wave functions . $E_\delta(N^e)$ is the eigen energy of the edge quasiparticle excitations and $E_{g,b}$ is the bulk state contribution to the ground state energy.

For a disc sample, the edge CFs are restricted in a circular strip near the boundary with its width $\delta r_i << R$ where R is the radius of the disc. Suppose we have N_e CFs at the edge, while the remaining $N - N_e$ are localized inside the bulk. Since we are only interested in low energy excitations of the quantum Hall liquid, and the bulk CFs contribute no gapless elementary excitations, we can focus our attention on the edge CFs by separating them from the bulk in Hamiltonian H_c, and treating the interactions with bulk ones as an 'external potential' which includes both vector potential and scalar one. The former supplies, in the mean-field approximation, an additional magnetic field that on average reduces the external magnetic field from B to B^*, that is

$$\begin{aligned} E_i^e &= \left\langle \sum_{j\in bulk} {}' E_{ij} \right\rangle_{bulk} + \sum_{j\in edge} {}' E_{ij} - \beta r_i^2 \\ &= \sum_{j\in edge} {}' E_{ij} - \beta^* r_i^2 \end{aligned}$$

where $\beta^* = \frac{|B^*|}{4}$ and $B^* = B/m$.

The scalar potential can partly screen the Coulomb interaction between the edge particles. If the magnetic field is absent, the screening is caused by 'mirror positive charges' that provide the scalar potential to slowly moving edge CFs so as to reduce the 1/r interaction to one with shorter range like dipole-dipole interaction. However, the effective magnetic field forces the CF's into cyclotron motion and the screening may be weakened. Even so, we still expect the bare Coulomb interaction between the edge CFs to be dramatically renormalized or screened to a shorter range interaction by considering the effect of the experimental devices such as a metal electrode. We represent the modified interaction potential as $V_{\mathrm{sc}}(\vec{r}_i - \vec{r}_j)$ and the modified form of the edge potential as $U_{\mathrm{eff}}(\vec{r}_i)$. Then the Hamiltonian for edge particles can be written as follows,

$$\begin{aligned} H_{edge} = \sum_{i=1}^{N_e} \Bigg[&\left(\frac{-i}{r_i}\frac{\partial}{\partial\phi_i} + \frac{\partial E_i^e}{\partial r_i}\right)^2 \qquad (2.16) \\ &+ \left(-i\frac{\partial}{\partial r_i} - \frac{1}{r_i}\frac{\partial E_i^e}{\partial \phi_i}\right)^2 - \frac{1}{r_i}\frac{\partial}{\partial r_i} + \frac{i}{r_i^2}\frac{\partial E_i^e}{\partial\phi_i}\Bigg] \\ &+ \sum_{i<j} V_{sc}(\vec{r}_i - \vec{r}_j) + \sum_i U_{\mathrm{eff}}(\vec{r}_i). \end{aligned}$$

By applying similar procedure to the edge Hamiltonian H_{edge} , we arrive at the same Hamiltonian as H' except that only edge electrons are included and U, V are replaced by their modified forms.

Under the condition $\delta r_i << R$, the transformed Hamiltonian can be reduced to the following one dimensional form,

$$\begin{aligned} H''_{1d} = &-\frac{1}{R^2}\sum_i \frac{\partial^2}{\partial\phi_i^2} \qquad (2.17) \\ &- \frac{m}{R^2}\sum_{i<j} \cot\frac{\phi_i - \phi_j}{2}\left(\frac{\partial}{\partial\phi_i} - \frac{\partial}{\partial\phi_j}\right) \\ &- 2K_0 \sum_i \left(\frac{-i}{R}\frac{\partial}{\partial\phi_i}\right) + V_{sc} + U_{eff} \end{aligned}$$

by making the substitutions $z_i = Re^{i\phi_i}$ where

$$K_0 = \frac{|B^*|R}{2} - \frac{(N_e - 1)m}{2R}. \qquad (2.18)$$

Then after a simple gauge transformation, which is equivalent to a translation of momentum along the edge that reads

$$-i\frac{\partial}{\partial\phi_i} \to -i\frac{\partial}{\partial\phi_i} + K_0, \qquad (2.19)$$

the linear term is cancelled, and we get

$$\begin{aligned} H''_{1d} = &-\frac{1}{R^2}\sum_i \frac{\partial^2}{\partial\phi_i^2} - \frac{m}{R^2}\sum_{i<j}\cot\frac{\phi_i-\phi_j}{2} \qquad (2.20) \\ &\times \left(\frac{\partial}{\partial\phi_i} - \frac{\partial}{\partial\phi_j}\right) + V_{sc} + U_{eff} \end{aligned}$$

We argue that the above Hamiltonian is in fact equivalent to the corresponding one describing the excitations of CSM if we switch off V_{sc} and U_{eff}. To see it clearly, we can follow a similar procedure to take a transformation of the CSM Hamiltonian.

$$H_{cs} = \sum_i -\frac{\partial^2}{\partial\phi_i^2} + \frac{m(m-1)}{4}\sum_{i<j}\left[\sin\left(\frac{\phi_{ij}}{2}\right)\right]^{-2} \qquad (2.21)$$

We separate the total wave function to a product of its ground state wave function Ψ_{csg} and an extra part Φ' which contains all the information of the excitations of CSM,

$$\Psi_{cs} = \Psi_{csg}\Phi' \qquad (2.22)$$

Therefore we get an effective Hamiltonian with the same form as H''_{1d} if we set $R = 1$,

$$\begin{aligned} H''_{cs} &= \Psi_{csg}^{-1} H_{cs} \Psi_{csg} \qquad (2.23) \\ &= -\sum_i \frac{\partial^2}{\partial\phi_i^2} - m\sum_{i<j}\cot\frac{\phi_i-\phi_j}{2}\left(\frac{\partial}{\partial\phi_i} - \frac{\partial}{\partial\phi_j}\right) \end{aligned}$$

Moreover if we set the ground state wave function of H_c to its one-dimensional limit, we arrive at exactly the ground state wave function of CSM.

$$\Psi_{csg}(\phi_1, ...\phi_{N_e}) = \prod_{i<j}\left[\sin\frac{\phi_{ij}}{2}\right]^m \qquad (2.24)$$

From the above derivation one can see that the $\nu = 1/m$ FQHE edge states are described by the CSM if we switch off all the interactions between CFs. Since the CSM is exactly solvable and its excitation wave functions can be calculated by using Jack's polynomials [25], the study of FQHE edge states will pose no difficulty to us if we can properly handle the influence of the interactions between edge particles. It will be discussed in the coming sections on what roles the interactions play in the description of FQHE edge states . One can see that the CSM has two-branches of the gapless excitations while the edge excitations of FQHE are chiral. This point will be clarified in Section IV.

III. NON-RENORMALIZATION OF THE EXPONENTS

It is well-known that the CSM is an example of one dimensional ideal excluson gas(IEG) [20] with the statistical parameter m. And the IEG is proved to describe the fixed point of the Luttinger liquid. The bosonization of CSM shows that the low-lying excitations are governed by a $c = 1$ CFT with the compactified radius $1/\sqrt{m}$ [19,20]. In this section, we would like to show that at least some kinds of short-range interactions between the CFs do not renormalize the topological exponent $g = m$ under the condition that the scatterings with large momentum transfer (including backward scattering and umklapp scattering) are absent because of the chirality.

To deal with the CSM with interactions, we begin with the asymptotic Bethe ansatz (ABA) equation [8],

$$k_i L = 2\pi I_i + \sum_j{}' \theta(k_i - k_j), \qquad (3.1)$$

where k_i is the pseudomomentum of particle i, L is the size of the one dimensional system concerned, and I_i gives the corresponding quantum number, which is an integer or half-odd. $\theta(k)$ represents the phase shift of a particle after a single collision with a pseudomomentum transfer of k. It has been proved that ABA equations give exact solutions to the energy spectrum of the CSM. We assume this approach could be generalized to the situations of CSM plus some other kind of interaction with force range shorter than $\frac{1}{r^2}$ potential in the sense of perturbation. This assumption is justified for the following reasons: First, at the edge of fractional quantum Hall liquid with $\nu = 1/m$, the linear density of the edge particles can be estimated as

$$\rho \propto n \times l_B \propto B^{-1/2}, \qquad (3.2)$$

where n is the average bulk density of the FQH liquid that is fixed and $l_B = eB/m^*c$ is the magnetic length corresponding to the magnetic field B. Under the condition of strong enough magnetic field, the edge particles can be regarded as a dilute one dimensional gas, where only two body collisions are important, and the free length between two collisions is long enough to allow the phase shift to reach its asymptotic value. Secondly, what we are concerned with is the property of low energy excitations near the Fermi surface, not the whole precise energy spectrum which can not be given by ABA. The low energy excitations involve only scattering processes with small momentum transfer Δk (because of Chirality, see Sec.IV), and are determined by the behavior of $\theta(k)$ around $k = 0$, which is dominated by $\theta_{cs}(k)$ (see below). Under the condition of low energy limit where we let Δk approach zero slowly, $\theta(k)$ will become asymptotically close to $\theta_{cs}(k)$, as a result we can expect ABA calculations to give asymptotically correct results. Here we implicitly assume : the low energy spectrum of the CSM varies continuously with respect to the addition of small perturbation without undertaking any abrupt changes like a phase transition. This assumption is reasonable, for the low energy limit of the CSM is the fixed point of Luttinger liquid which is robust against perturbations. Therefore, what follows from ABA, as we believe, is credible. As a matter of fact, for a large class of short-range interactions, we can expect the ABA works in describing the low-lying excitations of the system. Indeed, there are several kinds of short-range interactions whose low-lying excitations are governed by the ABA. An example of them is the $\delta^{(l)}$-function interaction with (l) representing the l-th derivative of the δ-function and l being restricted to $l < m$. The pseduopotentials used by Haldane [18] are other examples because of the vanishing of the expectation value of the pseudopotentials in the ground state.

To calculate the phase shift, we note the analog of the Schrodinger equation of the two-body CSM with an additional short-range interaction in the limit $L \to \infty$ to the radial equation of a three-dimensional scattering problem of a centrally symmetric potential. The topological exponent m corresponds to the total angular momentum l, i. e. $m = l + 1$. The Schrodinger equation reads

$$\frac{d^2\psi(x)}{dx^2} + \left[(E - V) - \frac{l(l+1)}{x^2}\right]\psi(x) = 0. \qquad (3.3)$$

The asymptotic solution of (3.3) for $x \gg 0$ is given by

$$\psi(x) \approx 2\sin(kx - \frac{1}{2}l\pi + \delta_l), \qquad (3.4)$$

where δ_l is the three-dimensional phase shift corresponding to the scattering potential V. In the sense of 1-d scattering,

$$\theta(k) = \pi(m-1)\mathrm{sgn}(k) - 2\delta_l(k). \qquad (3.5)$$

We see that the contribution of V to the phase shift is

$$\theta_{\mathrm{reg}}(k) = -2\delta_l(k), \qquad (3.6)$$

which is continuous and vanishing at $k=0$ if V is short-ranged (shorter than $1/r^2$).

Now, let's make the relation to the macroscopic theory. In terms of the partition function (2.16), there is a most probable edge CF number $\bar{N}^e$ which is given by $\delta Z/\delta N^e = 0$. $\bar{N}^e = \int dx \rho(x)$ with the edge density $\rho(x) = h(x)\rho_e$ [6]. Here $h(x)$ is the edge deformation and ρ_e is the average density of the bulk electrons. We do not distinguish $\bar{N}_e$ and N_e hereforth if there is no ambiguity. The low energy properties of the CSM can be obtained from the ABA equations,

$$\rho(k) = \rho_0(k) - \int_{-k_F}^{k_F} g(k-q)\rho(q)dq \tag{3.7}$$

$$\epsilon(k) = \epsilon_0(k) - \int_{-k_F}^{k_F} g(k-q)\epsilon(q)dq \tag{3.8}$$

where $k_F = \pi m N_0^e/L$,

$$g(k) = \frac{1}{2\pi}\frac{d\theta(k)}{dk}, \tag{3.9}$$

$\epsilon_0(k) = k^2 - k_F^2$ and $\rho_0(k) = \frac{1}{2\pi}$.

If we consider only the CSM without other interactions ,then we have

$$\theta_{cs}(k) = \pi(m-1)\mathrm{sgn}(k). \tag{3.10}$$

Substituting (3.10) into (3.8) and after the linearization, we get

$$\epsilon_{cs\pm}(k) = \begin{cases} \pm v_+(k \mp k_F), & \text{if } |k| > k_F \\ \pm v_-(k \mp k_F), & \text{if } |k| < k_F, \end{cases} \tag{3.11}$$

where

$$v_+ = \left.\frac{d\epsilon(k)}{dk}\right|_{k=k_F+0^+} = v_F \tag{3.12}$$

$$v_- = \left.\frac{d\epsilon(k)}{dk}\right|_{k=k_F-0^+} = \frac{v_F}{m}$$

with $v_F = 2k_F$ and

$$\rho_+ = \rho(k_F + 0^+) = \frac{L}{2\pi} \tag{3.13}$$

$$\rho_- = \rho(k_F - 0^+) = \frac{L}{2\pi m}$$

We rewrite the important equations essential to the bosonization for CSM as follows

$$v_+ = m v_- \tag{3.14}$$

$$\rho_+ = m\rho_-. \tag{3.15}$$

A successful bosonization of the theory with the refraction dispersion (3.11) has been done by the authors of [20] and one shows that the low-lying excitations of the CSM are controlled by the $c=1$ CFT with its compactified radius $\mathcal{R} = 1/\sqrt{m}$ [19]. This implies that the low-lying states of the CSM have the Luttinger liquid behaviors with the exponent $g=m$. We will be back to this issue later after we supplies the chiral constraint and then show that the edge states of FQHE have the CLL behaviors.

Now, let us see the effects of the interactions. Following our discussion that leads to the $c=1$ CFT with the compactified radius $\mathcal{R} = 1/\sqrt{m}$, the relations (3.15) are essential. We would like to check if they are renormalized by the interactions between CFs. Here, we limit our discussion to the case $m \neq 1$. We assume the ABA works to describe the low-lying excitations of the system with an additional short range interaction, which is consitent with the chirality of the edge excitations. Differentiating the phase shift (3.5) with respect to k, one has

$$g(k) = (m-1)\delta(k) + g_{\mathrm{reg}}(k). \tag{3.16}$$

The continuity of θ_{reg} implies that g_{reg} is no more singular than the δ-function at $k \to 0$. Therefore, we can prove that the relations (3.15) still hold even after we have introduced a short-range interaction. After differentiating the dressed energy equation (3.8) that is assumed holding for the short-range interaction we are using and in the dilute gas approximation, with respect to k, we obtain

$$\begin{aligned} v_\pm &= v_0 + \int_{-k_F}^{k_F} \epsilon(q)\frac{d}{dq} g(k_F \pm 0^+ - q)dq \\ &= v_0 - \int_{-k_F}^{k_F} \frac{d}{dq}\epsilon(q) g(k_F \pm 0^+ - q)dq \\ &+ \epsilon(k_F) g(k_F \pm 0^+ - k_F) \\ &- \epsilon(-k_F) g(k_F \pm 0^+ + k_F). \end{aligned} \tag{3.17}$$

The definition of k_F ,i.e., $\epsilon(\pm k_F) = 0$, leads to

$$\begin{aligned} v_+ - v_- &= \int_{-k_F}^{k_F} \frac{d}{dq}\epsilon(q)(m-1)\delta(q-k) \\ &= (m-1)v_-. \end{aligned} \tag{3.18}$$

Hence

$$v_+ = m v_-. \tag{3.19}$$

The value of $v_\pm$ can be modified by the interactions but the above relation does not change. Note that if $\epsilon(k_F) = 0$ and $\epsilon(-k_F)g(k_F \pm 0^+ + k_F)$ is continuous at k_F, the above conclusion still holds, which will be the case in the CLL derivation of Sec IV. By performing a similar procedure to $\rho(k)$, we can show (3.15) for $\rho_\pm$ as well. Therefore one can see that the bosonization process of the CSM is still applicable in the presence of perturbative

interactions, and the topological exponent $g = m$ is not renormalized by the short-range interaction. As a result, the compactified radius of the $c = 1$ CFT which governs the low-lying excitations of the theory does not change.

Let us give more comments on the conclusion drawn above. This result seems remarkable at the first sight, when compared with the standard Luttinger liquid theory, in which we will have the characteristic exponent renormalized once a short-range perturbative interaction is switched on. In fact, no inconsistences exist here. In the bosonization of the general Luttinger liquid, only short range interactions are considered, whose Fourier transformation $V(k)$ at k=0 possesses no singularity. Even if the divergence of $V(k)$ as k approaches zero does show up, it is suppressed by introducing something like a short-range cutoff or a long-range cutoff which makes the problem concerned more subtle. The exponent so obtained may be cutoff-dependent. So we can not naively apply it here. In contrast to the standard approach, the bosonization of the CSM is based on the especially simple form of the phase shift function of the $1/r^2$ interaction that is essential to the solution of ABA. The singularity here manifests itself as a step discontinuity which can be handled easily (no cutoff is needed). Because of the critical property of the $1/r^2$ interaction, no other interactions with shorter ranges can alter this discontinuity, which guarantees the robustness of the bosonization process. In short, the bosonization of CSM is not so general as the standard one, but it surely makes a step forward in understanding the low energy physics of nontrivial interactions.

We emphasize once again that both $1/r^2$ interaction and the chirality contribute to the robustness of $g = 1/m$ when $m > 1$. In general, the critical exponent will be changed by the introduction of other short-range interactions if the chirality is not present and backward scattering is allowed. In contrast, in case of $m = 1$, where we are actually dealing with a Fermi liquid, the discontinuity of the phase shift $\theta(k)$ is absent. So the above argument of robustness fails. An simple example is to consider a δ-function interaction. For $m > 1$, the short-range divergence of the $1/r^2$ potential requires that the wave function vanishes when two particles approach each other. Hence the δ-function contribution to the phase shift is completely suppressed in case of $m > 1$,while it does show up for $m = 1$ [26]. On the occasion of $m = 1$, however, the chirality alone serves as the determinant factor to ensure the non-renormalizability of $g = 1$, by prohibiting the left-right scattering part of perturbative interactions from modifying g . Therefore, one can see that the different microscopic mechanisms for $m > 1$ and $m = 1$ give the same macroscopic result.

From the above arguments, we see that the topological exponent is invariant to the perturbations introduced by additional interactions between particles, if their interaction range is shorter than that of $1/r^2$. However, the long range nature of Coulomb interaction allows it to dominate the $1/r^2$ interaction which gives $g = \nu$. Considering its especially singular behavior at k=0, we believe that the so called topological index can no longer survive, if an unscreened Coulomb interaction without any cutoff really exists. Fortunately, we have several possibilities that will lead to partial screening of the Coulomb interaction. In real experiments, the edge electrons actually are not isolated to a wire-like structure. There are bulk electrons adjacent to them, which can provide mirror charges and reduce the original Coulomb interaction to a shorter range interaction. What is more, metal electrodes commonly used in experiments to supply a confinement potential can also serve as a mirror charges provider. So we only have to concern ourselves with partly screened or drastically renormalized Coulomb interaction instead of the bare one. The effect of short-range interactions has been discussed in this section. We will consider the case where the Coulomb interaction is drastically renormalized later after we explain the chirality of the edge states.

IV. CHIRAL LUTTINGER LIQUID: THE MICROSCOPIC POINT OF VIEW

A. Microscopic Derivation of CLL from the Radial Equation

In the previous two sections, we freeze the radial degree of freedom of the edge particles and see that the azimuthal dynamics is described by the CSM. However, there are two branches of gapless excitations in the CSM and the chirality of the edge excitations are not shown. To arrive at the conclusion of chirality, we take the radial degree of freedom into account. Let us first make some simplifications before going into details. First, the interactions between CFs are assumed to be independent of the radial degree of freedom because of the small width of the edge. Moreover, we can think of the interaction between the CFs as consisting of only the $1/x^2$-type since we have demonstrated that short-range interactions do not renormalize the topological exponent $g = m$. At the end of this section, we will give a discussion about the effect of Coulomb interaction when it is drastically renormalized. Secondly, the edge potential is assumed as follows,

$$U_{\text{eff}}(r) = \begin{cases} \infty, & \text{if } r \geq R, \\ eE(r - R'), & \text{if } R' < r < R, \\ 0, & \text{if } r < R' \end{cases} \tag{4.1}$$

Here E is the value of the external field and one takes $v_d^* = cE/|B^*| > R_c^* \omega_c^*$. The exact value of R' is not important. The only requirement is $R' \simeq R - R_c^*$. In Sec. II, we take the approximation in the edge Hamiltonian (2.16) with $r_i \simeq R$ and arrive at the CSM. To consider the radial degree of freedom, we separate the wave function into the azumithal and radial parts near the edge,

$$\Psi_c(z_1, ..., z_{N_e}) = g(r_1, ..., r_{N_e}) \Psi_{cs}(\varphi_1, ..., \varphi_{N_e}), \tag{4.2}$$

where Ψ_{cs} is the eigenstate wave function of the CSM, which is antisymmetric and g is a symmetric radial wave function. Substituting the separated form of the wave function to the Schrodinger equation corresponding to H_{edge} (2.16), one finds that the equation can be reduced to the radial eigenstate equation which reads

$$\sum_i\left[-\frac{\partial^2}{\partial r_i^2}+(\frac{n'}{r_i}-\frac{|B^*|}{2}r_i)^2\right]g(r_1,...,r_{N_e}) \tag{4.3}$$
$$+[U_{eff}+O(\delta r_i/R)]g(r_1,...,r_{N_e})=Eg(r_1,...,r_{N_e}).$$

where $n'=n+\frac{m}{2}(N_e-1)$ and the terms $\frac{1}{R}\frac{\partial}{\partial r_i}$ have been absorbed into g by a simple transformation like the multiplication of $e^{\sum r_i/R}$. One can see that the radial eigenstate equation can be treated in the single particle picture except that the pseudomomenta $k=nR$ are related to one another by the ABA equations(3.1). It is reasonable to arrive at such a result because the interactions between CFs are the functions of $\vec{r_i}-\vec{r_j}$ and the radius-dependent part of the interactions is of order $\delta r/R$. Now we employ the harmonic approximation used by Halperin in the case of IQHE edge states [1]. Let us first turn off the applied electric field. The radial single particle wave equation in the stripe approximation reads

$$-\frac{d^2g}{dy^2}+B^{*2}y^2g=\varepsilon_+g, \tag{4.4}$$

for $n'>0$ and

$$-\frac{d^2g}{dy^2}+B^{*2}y^2g+|n'B^*|g=\varepsilon_-g, \tag{4.5}$$

for $n'<0$. Here $y=r-R_{n'}$ and

$$R_{n'}=\sqrt{\frac{2|n'|}{|B^*|}}. \tag{4.6}$$

Comparing (4.4) with (4.5), we see that the magnetic field separates the $n'<0$ sector from the $n'>0$ sector by an energy gap $|n'|\hbar\omega_c^*$. Therefore only the $n'>0$ (or equivalently, $k>-k_F$) sector needs to be considered for the low-lying excitations. This is the first sign of chirality. The harmonic equation (4.4) has its eigenstate energy

$$\varepsilon_{+,\nu^*}=\hbar\omega_c^*((\nu^*-1)+\frac{1}{2}), \tag{4.7}$$

if the center of the harmonic potential $R_{n'}\ll R$. This is consistent with the mean-field approximation to the bulk state because $R_{n'}\ll R$ actually corresponds to the bulk state of the theory if we recognize that the width of the harmonic oscillator wave function is about several times the cyclotron motion radius R_c^*. Since $R_{n'}$ is the function of n', $p_n=m^*\omega^*R_{n'}$ can be regarded as a momentum-like quantity. The harmonic oscillator energy for $R-R_{n'}\ll R_c^*$ implies that there is no left-side Fermi point. This provides a necessary condition of the chirality. To justify the CLL, one should show the existence of gapless excitations on the right side. It is known that the eigenstate energy at $R_{n'}=R$ is raised to

$$\varepsilon_{R,\nu^*}=\hbar\omega_c^*\left(2(\nu^*-1)+\frac{3}{2}\right), \tag{4.8}$$

because of the vanishing of the wave function at $r=R$. One asks that what happens if $R_{n'}$ is slightly away from R. To see it clearly, we rewrite (4.4) as

$$-\frac{d^2g}{d\tilde{y}^2}+B^{*2}\tilde{y}^2g+2B^{*2}(R-R_{n'})\tilde{y}g$$
$$+B^{*2}(R_{n'}-R)^2g=\varepsilon_+g, \tag{4.9}$$

where $\tilde{y}=r-R$. If $R_{n'}$ is very close to R, i. e., $|R-R_{n'}|\le R_c^*$, one can take the third term as perturbation and a first order perturbative calculation shows

$$\delta\varepsilon_{0,\nu^*}=\varepsilon_{+,\nu^*}-\varepsilon_{R,\nu^*}=v_c^*(p_n-p_R)+(p_n-p_R)^2, \tag{4.10}$$

where $v_c^*=2\pi^{-1/2}l_{B^*}\omega_c^*$ is of the order of the cyclotron velocity of the CF corresponding to B^* and then of the order of v_F. So, if we take $v_c^*\approx v_F$ and note that $p_n-p_R\approx k-K_0$, the dispersion (4.10) can be simply rewritten as

$$\delta\varepsilon_0(k)\approx(k-K_0+k_F)^2-k_F^2, \tag{4.11}$$

where k is given by the ABA equations (3.1). Near the Fermi point K_0, the dispersion can be linearized as

$$\delta\varepsilon_0(k)=v_F(k-K_0), \tag{4.12}$$

which implies that there is a right-moving sound wave excitation along the edge with the sound velocity v_F. There is another Fermi point $k=K_0-2k_F$, which corresponds to $R_{n'}\approx R-2R_c^*$ and is outside of the region we are considering and in fact belongs to the bulk state. To show that the above chiral theory has a Luttinger liquid behavior, we extend continuously $\delta\varepsilon_0(k)$ to all possible pseudomomenta which obey the ABA (3.1). Then, the equation (4.11) and (3.1) mean that the system is an IEG [20]. The problem can be solved by using a bosonization procedure developed in ref. [20]. The edge excitations can be obtained by considering only the properties of such an IEG system near $k\sim K_0$. Consequently, the low-lying excitations of the theory are controlled by the $c=1$ CFT with its compactified radius $\mathcal{R}=1/\sqrt{m}$ as we point out in the discussion of the CSM in Sec. III. However, the relevant excitations of the edge states include only the right-moving branch. In other words, the edge states are chiral and the sound wave excitations correspond to the non-zero modes of the right-moving sector of the $c=1$ CFT. There are two other kinds of edge excitations which correspond to the particle additions to the ground state and the current excitations along the edge respectively. The velocity relations of these excitations are given by [20]

$$v_M = mv_F, v_J = v_F/m, v_F = \sqrt{v_M v_J}. \tag{4.13}$$

The relations resemble those of Haldane's Luttinger liquid if one identifies m with the characteristic parameter $e^{-2\varphi}$ in the Luttinger liquid theory [28]. These observations are crucial to the conclusion that the edge states are controlled by the $c = 1$ CFT with its compactified radius $\mathcal{R} = 1/\sqrt{m}$.

To arrive at the effective theory of CLL, let us perform the following bosonization procedure.

According to (4.11) and (3.1), the edge excitations with the pseudomomentum k have their dressed energy

$$\varepsilon(k) = \begin{cases} (k^2 - k_F^2)/m, & |k| < k_F, \\ k^2 - k_F^2, & |k| > k_F. \end{cases} \tag{4.14}$$

Here we have made a translation $k \to k + K_0 - k_F$. The linearization approximation of the dressed energy near $k \sim \pm k_F$ is given by (3.11). In terms of the linearized dressed energy, we obtain a free fermion-like representation of the theory and then can easily bosonize it [20]. The Fourier transformation of the right-moving density operator is given by

$$\rho_q^{(+)} = \sum_{k>k_F} : c_{k-q}^\dagger c_k : + \sum_{k<k_F - mq} : c_{k+mq}^\dagger c_k : \tag{4.15}$$

$$+ \sum_{k_F - mq<k<k_F} : c^\dagger_{\frac{k-k_F}{m}+k_F+q} c_k :, \tag{4.16}$$

for $q > 0$ is the sound wave vector. Here c_k is a fermion annihilation operator. And a similar $\rho_q^{(-)}$ can be defined near $k \sim -k_F$. The bosonized Hamiltonian is given by

$$H_B = v_F \{ \sum_{q>0} q(b_q^\dagger b_q + \tilde{b}_q^\dagger \tilde{b}_q) + \frac{1}{2}\frac{\pi}{L}[mM^2 + \frac{1}{m}J^2]\}. \tag{4.17}$$

Thus, we have a current algebra like

$$[\rho_q^{(\pm)}, \rho_q^{(\pm)\dagger}] = \frac{L}{2\pi} q\, \delta_{q,q'}, \quad [H_B, \rho_q^{(\pm)}] = \pm v_F q \rho_q^{(\pm)}. \tag{4.18}$$

In the coordinate-space formulation, the normalized density field $\rho(x)$ is given by $\rho(x) = \rho_R(x) + \rho_L(x)$:

$$\rho_L(x) = \frac{M}{2L} + \sum_{q>0} \sqrt{q/2\pi L m}(e^{iqx} b_q + e^{-iqx} b_q^\dagger), \tag{4.19}$$

and $\rho_R(x)$ is similarly constructed from $\tilde{b}_q$ and $\tilde{b}_q^\dagger$. Here $b_q = \sqrt{2\pi/qL}\rho_q^{(+)\dagger}$ and so on. The boson field $\phi(x)$, which is conjugated to $\rho(x)$ and satisfies $[\phi(x), \rho(x')] = i\delta(x - x')$, is $\phi(x) = \phi_R(x) + \phi_L(x)$ with

$$\phi_L(x) = \frac{\phi_{0,L}}{2} + \frac{\pi J x}{2L} + i \sum_{q>0} \sqrt{\pi m/2qL}(e^{iqx} b_q - e^{-iqx} b_q^\dagger),$$

and a similar $\phi_L(x)$. Here M and J are operators with integer eigenvalues, and $\phi_0 = \phi_{l0} + \phi_{r0}$ is an angular variable conjugated to M: $[\phi_0, M] = i$. The Hamiltonian (4.17) becomes

$$H_B = \frac{v_F}{2\pi} \int_0^L dx\, [\Pi(x)^2 + (\partial_x X(x))^2], \tag{4.20}$$

where $\Pi(x) = \pi m^{1/2} \rho(x)$ and $X(x) = m^{-1/2}\phi(x)$. With $X(x,t) = e^{iHt} X(x) e^{-iHt}$, the Lagrangian density reads

$$\mathcal{L} = \frac{v_F}{2\pi}\, \partial_\alpha X(x,t)\, \partial^\alpha X(x,t). \tag{4.21}$$

We recognize that $\mathcal{L}$ is the Lagrangian of a $c = 1$ CFT. Since ϕ_0 is an angular variable, there is a hidden invariance in the theory under $\phi \to \phi + 2\pi$. The field X is thus said to be "compactified" on a circle, with a radius that is determined by the exclusion statistics:

$$X \sim X + 2\pi\mathcal{R}, \quad \mathcal{R}^2 = 1/m. \tag{4.22}$$

States $V[X]|0\rangle$ or operators $V[X]$ are allowed only if they respect this invariance, so quantum numbers of quasiparticles are strongly constrained.

In the present case, only the right-moving sector is relevant. So, we have an 'almost' chiral edge state theory whose sound wave excitation is chiral but there are charge leakages between the bulk and the edge. The leakages are reflected in the zero-mode particle number and current excitations [29]. In this almost chiral theory, the charge-one fermion operator is defined by

$$\Psi_R^\dagger(x) = \sum_{l=-\infty}^{\infty} \exp(i2(l + \frac{1}{2}m)\theta_R(x)) \exp(i\phi_R(x)), \tag{4.23}$$

where

$$\theta_R(x) = \pi \int_{-\infty}^{x} \rho_R(x') dx'. \tag{4.24}$$

The correlation function, then, can be calculated [25]

$$< \Psi_R^\dagger(x,t)\Psi_R(0,0) > \; = \sum_{l=-\infty}^{\infty} C_l \Big(\frac{1}{x - v_F t}\Big)^{(l+m)^2/m} \exp(i(2\pi(l + \frac{1}{2})x/L)). \tag{4.25}$$

The $l = 0$ sector recovers Wen's result [6]. In other words, the present theory justifies microscopically Wen's suggestion of CLL of the FQHE edge states .

Now, let us turn on the electric field. The external scalar potential perturbation is added to the harmonic oscillator equation (4.9)

$$-\frac{d^2 g}{d\tilde{y}^2} + B^{*2}\tilde{y}^2 g + 2B^{*2}(R - R_{n'})\tilde{y} g$$
$$+ B^{*2}(R_{n'} - R)^2 g + eE(r - R')g = \varepsilon_+ g. \tag{4.26}$$

Up to a zero position shift, (4.26) yields a harmonic potential centered at R perturbed by

$$(2B^{*2}(R-R_{n'})+eE)\tilde{y}. \tag{4.27}$$

A perturbative calculation to the second order shows

$$\delta\varepsilon_{d,0}(k)\approx(k-K_0+k_F^*)^2-k_F^{*2}, \tag{4.28}$$

where $k_F^*=\frac{1}{2}v_d^*+k_F$ with $v_d^*=cE/|B^*|$, the drift velocity of a CF in the effective field B^*. So, (4.28) and (3.1) define an new IEG system. All discussions made above for the situation with no electric field are valid after a replacement of k_F by k_F^*. One notes that $v_F\ll v_d^*$. So, $v_F^*=2k_F^*\approx v_d^*$. The current velocity $v_J=v_d$ is related to the sound wave velocity $v_s\approx v_d^*$ by Haldane's velocity relation $v_J=v_s/m$.

B. Further Discussion of Coulomb Interaction

In this subsection, we give a further discussion of the effects of the interactions between CFs . The short-range interactions do not renormalize the topological exponent $g=m$ because the conclusion drawn from Sec. III is also applicable to the last subsection. Here what interests us is the Coulomb interaction.

The wave equation of the one-dimensional Hamiltonian (2.21) with an additional Coulomb interaction (4.29) for $N_e=2$ can be exactly solved in the large L limit.

$$V_{\rm coul}=\frac{\alpha\pi}{L|\sin(\pi x_{12}/L)|} \tag{4.29}$$

Actually, this 1d Hamiltonian is the same as the radial part of the Hamiltonian of an electron scattered by a three-dimensional Coulomb potential [30]. The phase shift, is then given by

$$\theta(k)={\rm sgn}(k)\tilde{\phi}\pi-2{\rm arg}\Gamma(m+i/k). \tag{4.30}$$

By going to the dilute gas limit, $x_1\ll x_2\ll ...\ll x_{N_e}$, the N_e body problem can be solved asymptotically by means of the Bethe ansatz with the trial wave function

$$\begin{aligned}\Phi_s(x_1,..,x_{N_e})&=\sum_P A(P)\exp\{i\sum_i k_{P_i}x_i \\ &-i\sum_{i<j}F(k_{P_i}-k_{P_j},x_i-x_j)\},\\ F(k,x)&=\frac{\alpha}{2k}\ln(\left[\tan\frac{\pi x}{2L}\right]\frac{2\pi k}{L}).\end{aligned} \tag{4.31}$$

The asymptotic Bethe ansatz equations are given by $n_i=I_i+\frac{1}{2\pi}\sum_{j\neq i}\theta(k_i-k_j)$. Then we assume the Coulomb interaction to be drastically renormalized so that $\alpha L\ll 1$, which means that the minimal pseudomomenta spacing $\delta k\gg\alpha$. In the thermodynamic limit at low temperature, the thermodynamic Bethe ansatz equation for $\rho(k)$ can be solved iteratively. By integrating $\rho(k)$ with respect to k, one has

$$k=\tilde{k}+C\ln(k-k_F), \tag{4.32}$$

where $\tilde{k}$ is regular when $k-k_F$ tends to zero and C is a constant in proportion to αL. Hence, the radial equation reads

$$\begin{aligned}&-\frac{d^2g}{dy^2}+\omega_c^{*2}y^2g \\ &+C'y\ln(p_n-p_R)g=\varepsilon g,\end{aligned} \tag{4.33}$$

where $y=r-R_{\tilde{n}}$. By taking the approximation $r\simeq R$ in the logarithmic term, one can see that the dispersion relation reads, after switching on the external electric field,

$$\delta\varepsilon_{\tilde{n},\nu^*}=(p_n-p_R)(v_d^*+A\ln(p_n-p_R)), \tag{4.34}$$

with A proportional to αL. We then obtain a branch of excitations with the dispersion $q\ln q$, which is precisely the 1d plasmon excitation caused by the Coulomb interaction [21].

V. EDGE STATES WITH OTHER ODD DENOMINATOR FILLING FACTORS

In the rest of the paper, we would like to generalize the approach developed in the previous sections to deal with the other odd denominator filling factors. In general, one can transform the electronic Hamiltonian to the CF Hamiltonian like

$$\begin{aligned}H_{\rm cf}&=\sum_{I=1}^{P}\sum_{i_I=1}^{N_I}\frac{1}{2m^*}\Big[-i\hbar\frac{\partial}{\partial\vec{r}_{i_I}}+\frac{e}{c}\vec{A}_{i_I} \\ &+\frac{e}{c}\vec{a}_{i_I}\Big]^2+\sum_I(V_I+U_I),\end{aligned} \tag{5.1}$$

where $I=1,...,P$ are the CF Landau level indices and $\sum N_I=N_e$. The statistic gauge field is defined by

$$\vec{a}_{i_I}=\phi_0\sum_{J=1}\sum_{j_J}{}'K_{IJ}\frac{\hat{z}\times(\vec{r}_{i_I}-\vec{r}_{j_J})}{|\vec{r}_{i_I}-\vec{r}_{j_J})|^2}, \tag{5.2}$$

where the prime symbol means that $j_J\neq i_I$ if $I=J$. The matrix K_{IJ} is symmetric with odd diagonal elements, which is introduced by Wen and Zee [24] to describe the effective field theory of the FQHE. By using a gauge invariance argument, Wen has pointed out that a positive eigenvalue corresponds to a branch of right-moving edge excitations while a negative one to a left-moving branch [6]. Here, we would like to give a microscopic justification of the macroscopic effective theory of the edge states. Applying the mean-field approximation to the bulk states, one can see that

$$\nu = \sum_{IJ} (K^{-1})_{IJ}. \tag{5.3}$$

For explicity, we consider the case of multi-layer FQHE. In this case, the electronic ground state wave function can be simply written out in the lowest Landau level version,

$$\Psi_e(z_1, ..., z_{N_e}) = \prod_{I=1}^{P} \prod_{i_I < j_I} (z_{i_I} - z_{j_I})^{K_{II}} \tag{5.4}$$
$$\prod_{I<J}^{P} \prod_{i_I, j_J} (z_{i_I} - z_{j_J})^{K_{IJ}}$$
$$\times \exp(-\frac{1}{4} \sum_{I, i_I} |z_{i_I}|^2).$$

By conducting the transformations similar to those we use in the case of $\nu = 1/m$, we arrive at a 1-d generalized CSM whose Hamiltonian reads,

$$H_{gcs} = \sum_{I, i_I} -\frac{\partial^2}{\partial x_{i_I}^2} + \frac{\pi^2}{2L^2} \sum_{I, J; i_I \neq j_J} g_{IJ} \left[\sin \frac{\pi(x_{i_I} - x_{j_J})}{L} \right]^{-2}. \tag{5.5}$$

The corresponding many-body problem is also exactly solvable and the corresponding Bethe ansatz equations read as follows

$$L k_{i_I} = 2\pi I_{i_I} + \sum_{J} \sum_{j_J}{}' (K_{IJ} - 1) \mathrm{sgn}(k_{i_I} - k_{j_J}). \tag{5.6}$$

The radial wave equation for a single particle is given by

$$-\frac{d^2 g_I}{dy_I^2} + B^{*2} (\frac{n'_I}{r_I} - \frac{|B^*|}{2} r_I)^2 g_I + |B^*| v_{Id}^* y_I g_I = \varepsilon_I g_I, \tag{5.7}$$

where $v_{Id}^* = cE_I/|B^*|$ and E_I is the applied electric field corresponding to the potential $U_I(r_I)$. Then, a procedure parallel to that discussed in the previous sections leads to p-branches of edge excitations with sound wave velocities $v_{IF} \approx v_{Id}^*$ for the I-th branch. This recovers the CLL theory [6]. Although our explicit derivation is applied to the multi-layer quantum Hall liquid, we believe that the theory also holds in the single-layer case because the explicit ground state wave function is not crucial. What is important is the Bethe ansatz equations (5.6), which as we believe are valid for the single-layer case although the explicit derivation needs more work.

VI. CONCLUSIONS

In conclusion, we have given the detailed derivation of a microscopic model of the CF-type excitations of the FQHE edge with odd denominator filling factor ν. We start with the general Hamiltonian for a two-dimensional system of interacting electrons in a strong perpendicular magnetic field and confined to some specific geometry like a disc. The potential at the edge is assumed to maintain a sharp shape. After a CF type of anyon transformation, we obtain a Hamiltonian whose ground state wave function can be simply written out. By extracting the excitation part from the wave function, we arrive at a form of Hamiltonian which is shown to be equivalent to the corresponding Hamiltonian of the CSM when reduced to one dimension. We then take into account the influences of the interactions between edge particles. We demonstrate that the bosonization process specific to CSM is not changed by the introduction of some kinds of short-range interactions. The characteristic exponent $g = 1/m$ is indeed robust against perturbations under the condition of chirality. The low-lying excitations of the CF system are governed by a $c = 1$ CFT with its compactified radius $\mathcal{R} = 1/\sqrt{m}$ for the case of $\nu = 1/m$. By taking the radial degree of freedom into account, we show that the microscopic theory indeed justifies the macroscopic CLL. We also generalize the approach applied to $\nu = 1/m$ to other cases with odd denominator filling factors.

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